RELAXATION OF TUBES AND BUCKLING OF BARS MADE OF VISCOPLASTIC MATERIALS

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The behavior of two structural elements made of ideally plastic material with nonlinear viscosity is investigated.

This model was first proposed by Odqvist [1] and employed by Rozenblyum [2]. Odqvist's model received satisfactory experimental confirmation in the work of Gardner and Miller [3], who clearly observed the yield point; (up to a certain stress level there is a nonlinear relation between the steady-state creep stresses and rates, and beyond a critical stress ("creep limit") flow at arbitrary strain rates is observed).

In a number of cases neglecting the elastic strains leads to too rough an estimation of the real behavior of structural elements. Thus, an investigation of the process of stress relaxation must take instantaneous elasticity into account.

The first part of this paper is concerned with the problem of stress relaxation in a tube fitted over a rigid shaft when the tube material obeys the following conditions (Fig. 1a, b). Everywhere where the yidld point has not been reached the strains are elastic; moreover, at stresses above a certain value creep strains (steady-state or transient finite) develop. The yield point is the maximum permissible stress for the material; in regions where the yield point is reached the elastic and creep strains may be neglected as compared with the plastic strains.

The second part of the paper is devoted to the buckling of bars under conditions of nonlinear creep.

As distinct from [4-6], in which a bar with an idealized cross section is investigated, we will consider a solid bar and take into account the variability of the stresses over the section. Moreover, at stresses less than σ_s the steady-state creep relations are taken without allowance for instantaneous elastic strains—in fact, the model employed is that of a rigid, perfectly plastic medium with instantaneous deformation allowing for nonlinear viscosity.

§1. Initial state of stress and strain in a pressurized tube. We will consider an incompressible tube of circular cylindrical cross section made of an elastic-perfectly plastic material under conditions of plane deformation. The dimensionless radii of the tube are: inside—a, outside—1.

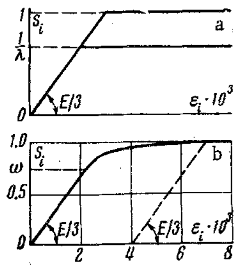

Fig. 1

In the presence of a uniform internal pressure p in the elastic state the stress components take the form [7]

$$\sigma_r = -p^*\left(\frac{1}{r^2}-1\right), \quad \sigma_\theta = p^*\left(\frac{1}{r^2}+1\right), \quad \sigma_z = \frac{\sigma_r+\sigma_\theta}{2},$$
$$\tau_{r\theta} = \tau_{\theta z} = \tau_{zr} = 0, \qquad p^* = \frac{pa^2}{(1-a^2)}. \tag{1.1}$$

The stress intensity

$$\sigma_i = \frac{\sigma_\theta - \sigma_r}{2} = \frac{p^*}{r^2}. \tag{1.2}$$

For an incompressible tube the relation between radial displacement and radius u(r) takes the form

$$u(r) = 1.5\, p^* E^{-1} r^{-1},$$

where E is the modulus of elasticity of the material. In accordance with (1.2), the stress intensity reaches a maximum at the inside surface of the tube. If σ_i is such that $\sigma_i = q$ at $r = a$, where q is the limiting value of σ_i, then a plastic zone develops at the inside surface. The dimensionless radius of the interface between the elastic and plastic zones will be denoted by c.

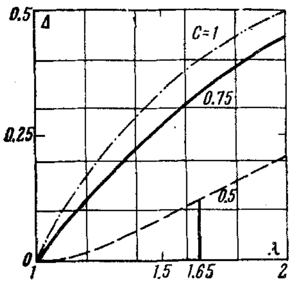

Fig. 2

In the plastic zone

$$\sigma_r = -p + 2q \ln\frac{r}{a}, \qquad \sigma_\theta = -p + 2q\left(1 + \ln\frac{r}{a}\right),$$
$$\sigma_i = q \quad (a \leqslant r \leqslant c). \tag{1.3}$$

In the elastic zone

$$\sigma_r = -qc^2(r^{-2}-1), \qquad \sigma_\theta = qc^2(r^{-2}+1),$$
$$\sigma_i = qc^2 r^{-2} \quad (c \leqslant r \leqslant 1). \tag{1.4}$$

The radius of the interface between zones is related with the internal pressure as follows:

$$0.5\frac{p}{q} = \ln\left(\frac{c}{a}\right) + 0.5(1-c^2).$$

The radial displacements u(r) of the tube in the elastoplastic state have the following dependence on r:

$$u(r) = 1.5\, qc^2 E^{-1} r^{-1}.$$

§2. Solution of problem with allowance for steady-state creep. This model is an attempt to describe the processes in the tube after it has been dynamically fitted over a rigid shaft. In simple tension (or pure torsion) diagrams the Young's modulus (or shear modulus) remains unchanged at different strain rates, whereas the yield point rises with increase in strain rate. The tube is fitted in the cold state at such a speed that the yield point in shear increases from the static value k to the dynamic value q: $q/k = \lambda > 1$. We introduce the dimensionless stresses $s = \sigma/q$. If $s_i \leq 1/\lambda$ everywhere at $a \leq r \leq 1$, then the tube remains plastic. This case is of no interest; therefore in what follows we will everywhere assume that the state of stress at the initial moment is such that on the inside part of the tube or throughout the tube $s_i > 1/\lambda$. In this region steady-state creep develops in accordance with the power law

$$(e_i^c)^{\cdot} = Bq^n(s_i - 1/\lambda)^n.$$

Here $(\varepsilon_i^c)^{\cdot}$ is the creep rate intensity, B and n are constants characterizing the creep process. As $t \to \infty$ (t is time) the material becomes elastic-perfectly plastic with a static yield point $s_i = 1/\lambda$ (Fig. 1a).

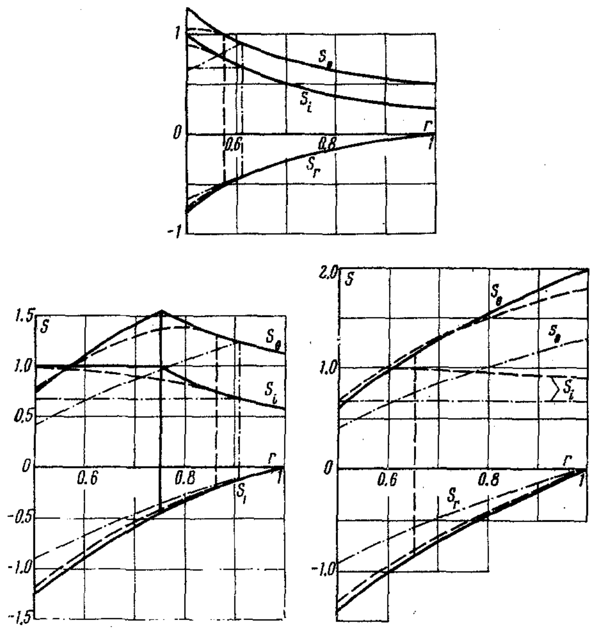

Fig. 3

At the initial moment the internal pressure p_0 imparts to the inside contour of the tube a displacement $u(a) = 1.5\, p^* E^{-1}$, if the entire tube is in the elastic state, or $u(a) = 1.5\, qc^2E^{-1}a^{-1}$, if the inside part $a \le r \le c$ is in the perfectly plastic state $s_i = 1$. In accordance with the formulation of the problem, $u(a)$ does not vary with time. From the conditions of plane deformation and incompressibility it follows that u(r) is also constant in time.

We will consider that part of the tube $a \le r \le d \le 1$ (or the whole tube), in which at the initial moment $s_i > 1/\lambda$. The relation between the stress and strain rate intensities

$$\varepsilon_i^{\cdot} = \frac{3q}{E} s_i^{\cdot} + Bq^n \left(s_i - \frac{1}{\lambda}\right)^n \tag{2.1}$$

and the condition $\partial u/\partial t = 0$ give the differential equation in s_i

$$\frac{3}{E} s_i^{\cdot} + Bq^{n-1}\left(s_i - \frac{1}{\lambda}\right)^n = 0.$$

Integrating, we obtain

$$s_i(r, t) = \frac{1}{\lambda}\left\{1 + (\lambda s_{i0} - 1)\left[1 + \frac{BEq^{n-1}(s_{i0} - 1/\lambda)^{n-1}}{3(n-1)} t\right]^{-\frac{1}{n-1}}\right\}. \tag{2.2}$$

The function $s_{i0} = s_{i0}(r)$—the distribution of s_i at $t = 0$—is found from (1.1) or (1.3)–(1.4). Equation (2.2) gives $s_i(r, t)$ at any point of the tube at any moment of time. After this it is possible to determine $s_r(r, t)$ from the equation

$$s_r(r, t) = \int_1^r \frac{2s_i(r, t)}{r} dr, \tag{2.3}$$

a consequence of the equilibrium equation, and then $s_\theta(r, t)$ from (1.2). We denote the stress components as $t \to \infty$ by $s_{i\infty}$, $s_{r\infty}$, $s_{\theta\infty}$.

We will determine the relative decrease in pressure $\Delta = (p_0 - p_\infty)/p_0$ for an elastic and elastoplastic distribution s_{i0}. Four cases need to be considered.

When $s_{i0} < 1$ the two cases

$$\Delta = 1 - \frac{k}{p_0}\left[1 + \ln\frac{p_0^*}{ka^2} - \frac{p_0^*}{k}\right] \qquad (ka^2 \leqslant p_0^* \leqslant k),$$

$$\Delta = 1 - \frac{2k}{p_0} \ln\frac{1}{a} \qquad (k \leqslant p_0^* \leqslant ka^2\lambda).$$

When $s_{i0}(r) = \begin{cases} 1 & (a \leqslant r \leqslant c) \\ c^2/r^2 & (c \leqslant r \leqslant 1) \end{cases}$ the two cases

$$\Delta = \frac{(\lambda - 1)[1 + 2\ln(c/a)] - \ln\lambda}{\lambda[2\ln(c/a) + 1 - c^2]} \qquad (a \leqslant c \leqslant \lambda^{-0.5}),$$

$$\Delta = 1 - \frac{2\ln(1/a)}{[2\ln(c/a) + 1 - c^2]\lambda} \qquad (\lambda^{-0.5} \leqslant c \leqslant 1).$$

Figure 2 presents $\Delta(\lambda)$ curves for a tube with $a = 0.5$ for limiting elastic ($c = 0.5$), elastoplastic ($c = 0.75$) and limiting plastic ($c = 1$) states at the initial moment. In Fig. 3a, b, c the continuous lines represent the relations $s_{i0}(r)$, $s_{r0}(r)$, $s_{\theta 0}(r)$, the chain-dotted lines the relations $s_{i\infty}(r)\, s_{r\infty}(r)$, $s_{\theta\infty}(r)$ at $\lambda = 1.5$ for $a = 0.5$, $c = 0.5$, 0.75, 1. At $c = 0.5$, $c = 0.75$, $c = 1$ we have $\Delta = 8.4\%$, 26.6%, 33.4%, respectively.

§3. Solution of the relaxation problem for a tube with allowance for transient finite creep. We will consider a tube fitted over a rigid shaft at a strain rate corresponding to an elastic-perfectly plastic $\sigma_i - \varepsilon_i$ diagram for the tube material with the same limiting value $\sigma_i = k$ as in the static $\sigma_i - \varepsilon_i$ diagram. In this case we refer all the stresses to k: $s = \sigma/k$. Let the static diagrams (Fig. 1b) at $0 \le s_i \le \omega < 1$ (where ω corresponds to the elastic limit) coincide with the dynamic diagram; at $\omega < s_i < 1$ the elastic strains of the dynamic diagram are added to the transient creep strains, while at $s_i = 1$ the material is perfectly plastic. The static diagram thus introduced agrees with experiments on titanium, mild steel and other materials.

We introduce the following relation between the intensities of stress, strain and the corresponding rates at $\omega < s_i < 1$:

$$\varepsilon_i^{\cdot} = \frac{3ks_i^{\cdot}}{E} + Bk^n(s_i - \omega)^n - A\left(\varepsilon_i - \frac{3ks_i}{E}\right). \tag{3.1}$$

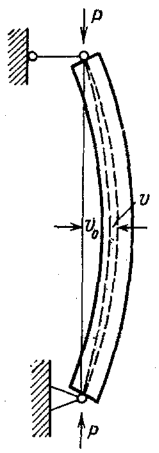

Fig. 4

Here the constants B and n characterize the steady-state creep, and the constant A the creep attenuation. At the initial moment $t = 0$ there follows from (3.1)

$$\varepsilon_i = 3ks_i/E, \tag{3.2}$$

which corresponds to the dynamic diagram. To obtain the static diagram we integrate (3.1) with the condition $s_i = 0$. We obtain

$$\varepsilon_i(t) = \frac{3ks_i}{E} + \frac{Bk^n}{A}(s_i - \omega)^n(1 - e^{-At}),$$

as $t \to \infty$

$$\varepsilon_i = \frac{3ks_i}{E} + \frac{Bk^n}{A}(s_i - \omega)^n. \tag{3.3}$$

At $s_i = \omega$ curves (3.2) and (3.3) merge smoothly with each other. At $s_i = 1$ the derivative $ds_i/d\varepsilon_i$ has a discontinuity, which can be eliminated at the expense of a complication of (3.1).

We now turn to the problem of relaxation of an incompressible tube. As in the preceding section, incompressibility gives $\varepsilon_i = \varepsilon_{i0} =$ = const, $\varepsilon_i{}^{\cdot}{}_0 = 0$. We integrate (3.1) under these conditions:

$$t = \frac{3k}{E}\int_{s_{i0}}^{s_i} \frac{ds_i}{A(\varepsilon_{i0} - 3ks_i/E) - Bk^n(s_i - \omega)^n}. \quad (3.4)$$

The distribution $s_{i0}(r)$ is determined on the basis of (1.1) or (1.3), (1.4), in which q must be replaced by k.

Equation (3.4) gives the distribution $s_i = s_i(r, t)$. Then, in accordance with (2.3) and (1.2), we can find the stresses $s_r(r, t)$ and $s_\theta(r, t)$ at any moment of time. Since in the general case Eq. (3.4) cannot be completely integrated, we performed calculations for a tube with $a = 1/2$, whose material remains elastic at $0 \le s_i \le 0.75$. Three different values of c were considered: 0.5, 0.75, and 1. As an example, we used the following values of the material characteristics: $n = 5$, $E/k = 10^3$, $Bk^n = 0.032\ \mathrm{hr}^{-1}$. The constant A was selected so that the curve (3.3) intersected the straight line $s_i = 1$ at a point corresponding to the nominal yield point and determined using $\varepsilon_s = 0.2\%$, i.e., $\varepsilon_{is} = 0.004$ (Fig. 1b), $A = 0.0078\ \mathrm{hr}^{-1}$.

The distributions $s_{i0}(r)$, $s_{r0}(r)$, $s_{\theta 0}(r)$ (Fig. 3a, b, c), calculated for the two models, coincide. The functions $s_{i\infty}(r)$, $s_{r\infty}(r)$, $s_{\infty 0}(r)$ are represented in Fig. 3a, b, c by a broken line. The relative decrease in the pressure of the tube on the shaft $\Delta = (p_0 - p_\infty)/p_0$ is 0.57% at c = 0.5, $\Delta = 3.42\%$ at c = 0.75 and $\Delta = 3.69\%$ at c = 1.

§4. **Buckling of a bar.** We will consider the problem of the buckling of a bar with an initial curvature under the action of an axial load. For the stress-strain relation we will take an expression of the type $\sigma = D\varepsilon^{\cdot m}$. We will consider a bar (Fig. 4) of constant cross section compressed by a longitudinal force P and having an initial curvature $v_0(x)$. During creep the deflection will increase, and the increase in deflection will be denoted by $v(x, t)$. We will adopt the hypothesis of plane sections; then, assuming that the deflections are small, we can write

$$\varepsilon = \varepsilon_0 + z\frac{\partial^2 v}{\partial x^2}, \quad (4.1)$$

where ε_0 is the deformation of the bar axis, and z is the coordinate of the section of the bar in the plane of bending. The equilibrium equations for the bar can be written in the form

$$P = \int_{-h}^{h} b\sigma dz, \qquad -P(v + v_0) = \int_{-h}^{h} b\sigma z dz, \quad (4.2)$$

where 2h is the height of the bar.

We will investigate the behavior of a hinged bar. In this case we assume that both the initial and the secondary deflections can be sufficiently accurately approximated by a single half-wave of a sinusoid:

$$v_0 = a_0 \sin\frac{\pi x}{L}, \qquad v = a \sin\frac{\pi x}{L}. \quad (4.3)$$

We satisfy Eqs. (4.2) by the collocation method. We introduce the dimensionless parameters (b_0 is the mean thickness of the bar) bar)

$$\frac{b}{b_0} = b_1, \quad P_1 = \frac{P_0}{Db_0h}, \quad \gamma = \frac{P}{2b_0h\sigma_s},$$
$$u = \frac{a}{h}, \quad \rho = \frac{\pi^2h^2}{L^2}, \quad \xi = \frac{z}{h}. \quad (4.4)$$

Using (4.1)-(4.4) for the case when there are no plastic regions over the cross section of the bar, we obtain

$$P_1 = \int_{-1}^{1} b_1(\varepsilon_0^{\cdot} - \rho u^{\cdot}\xi)^m d\xi, \quad -P_1(u + u_0) = \int_{-1}^{1} b_1(\varepsilon_0^{\cdot} - \rho u^{\cdot}\xi)^m \xi d\xi. \quad (4.5)$$

We make the substitution $\varepsilon_0^{\cdot} = k\rho u^{\cdot}$; then system (4.5) can be written in the form

$$P_1 = (\rho u^{\cdot})^m I_0, \qquad I_0(k) = \int_{-1}^{1} b_1(k - \xi)^m d\xi,$$
$$-P_1(u + u_0) = (\rho u^{\cdot})^m I_1, \qquad I_1(k) = \int_{-1}^{1} b_1(k - \xi)^{m+1}\xi d\xi. \quad (4.6)$$

From (4.6) it is easy to obtain a single equation for determining k as a function of time t:

$$\rho\left(-\frac{I_1}{I_0}\right)'_k k^{\cdot} = \left(\frac{P_1}{I_0}\right)^{1/m}. \quad (4.7)$$

For a bar of rectangular cross section ($b_1 = 1$) we have

$$I_0 = \frac{(k+1)^{m+1} - (k-1)^{m+1}}{m+1},$$
$$I_1 = \frac{k}{m+2} I_0 - \frac{(k+1)^{m+1} + (k-1)^{m+1}}{m+2}.$$

We derive the boundary conditions for k determined from (4.7). At t = 0 we have the condition u = 0 and hence from (4.6) we find $I_0(k_0)u_0 = -I_1(k_0)$.

The condition for the appearance of plasticity on the concave side of the bar gives an expression for finding the other boundary of the region of variation of k where equation (4.7) holds true.

From the condition $\sigma = \sigma_s$ at $\xi = -1$ we obtain

$$\sigma_s = A(\rho u^{\cdot})^m (k_1 + 1)^m. \quad (4.8)$$

Eliminating $u^{\cdot}$ from (4.6) and (4.8), we find the equation for k_1:

$$AP_1(k_1 + 1)^m = \sigma^s I_0(k_1), \text{ or } 2\gamma(k_1 + 1)^m = I_0(k_1). \quad (4.9)$$

When $\gamma < 1/r$ the solution of (4.9) can be represented with sufficient accuracy in the form $k_1 = \gamma/(1 - m\gamma)$; at γ close to 1 $k_1 \approx$ $\approx \gamma m/(1 - \gamma)$.

Thus, for the time of onset of plastic flow on the concave face of the bar τ_1 we obtain the expression

$$\int_0^{\tau_1} P_1^{1/m} dt = \rho\int_{k_0}^{k_1} I_0^{1/m}\left(-\frac{I_1}{I_0}\right)'_k dk. \quad (4.10)$$

For a bar of rectangular cross section expression (4.10) assumes the form

$$\int_0^{\tau_1} P_1^{1/m} dt = \frac{\rho}{(m+2)}\int_{k_1}^{k_0} f(k)\, dk,$$
$$f(k) = \left[\frac{(k+1)^{m+1} - (k-1)^{m+1}}{(m+1)}\right]^{1/m}\left\{1 - \frac{4(m+1)^2(k^2-1)^m}{[(k+1)^{m+1} - (k-1)^{m+1}]^2}\right\}.$$

We now turn to the case when a single plastic region propagates over the cross section of the bar. Then system (4.5) for a bar of rectangular cross section is replaced by the following equations:

$$2\gamma = \xi_1 + 1 + \frac{A}{\sigma_s}\int_{\xi_1}^{1}(\varepsilon_0^{\cdot} - u^{\cdot}\rho\xi)^m d\xi, \qquad \frac{A}{\sigma_s}(\varepsilon_0^{\cdot} - u^{\cdot}\rho\xi_1) = 1,$$
$$-2\gamma(u + u_0) = \frac{\xi_1^2 - 1}{2} + \frac{A}{\sigma_s}\int_{\xi_1}^{1}(\varepsilon_0^{\cdot} - u^{\cdot}\rho\xi)^m \xi\, d\xi. \quad (4.11)$$

Here ξ is the boundary between the viscous and plastic zones. This system characterizes a rectangular zone: it is real up to the moment when a plastic region appears on the convex face of the bar. The condition for the appearance of such a region has the form

$$A(\varepsilon_0^{\cdot} - u^{\cdot}\rho)^m = -\sigma_s. \quad (4.12)$$

We note that condition (4.12) gives the following values of the parameters:

$$k = \gamma, \qquad \xi_1 = 2\gamma - 1, \qquad u^{\cdot}\rho = \frac{1}{1-\gamma}\left(\frac{\sigma_s}{A}\right)^{1/m}. \tag{4.13}$$

Knowing $u^{\cdot}$ at the end of the first period and using the equation for $u^{\cdot}$ from (4.13), it is easy to estimate the duration of the second period by approximating the function u.

We will investigate the behavior of the bar when two plastic zones propagate over the cross section. We denote by ξ_2 the boundary between the zone of viscous flow and the second plastic region. We then have the two relations

$$A(\varepsilon_0^{\cdot} - \rho u^{\cdot}\xi_1)^m = \sigma_s, \qquad A(\varepsilon_0^{\cdot} - \rho u^{\cdot}\xi_2)^m = -\sigma_s. \tag{4.14}$$

By using (4.13) we can reduce the first equilibrium equations to the form

$$2\gamma = \frac{m}{(m+1)}(\xi_1 + \xi_2) + \frac{2\varepsilon_0}{u^{\cdot}\rho(m+1)}. \tag{4.15}$$

From (4.14) and (4.15) we obtain $\xi_1 + \xi_2 = 2\gamma, \varepsilon_0^{\cdot} = u^{\cdot}\rho\gamma$. Then, from the second equilibrium equation we find an equation for u:

$$(u^{\cdot}\rho)^2[1 - 2\gamma(u + u_0) - \gamma^2] = \frac{m}{(m+2)}\left(\frac{\sigma_s}{A}\right)^{2/m}. \tag{4.16}$$

The critical total deflection as $u^{\cdot} \to \infty$ is found from the relation

$$u^* + u_0 = (1 - \gamma^2)/2\gamma.$$

Equation (4.16) can easily be integrated for constant γ.

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